Please see Sections 1 and 3 of Tyler's notes if you need further assistance. This handout is intended to be a review.

Predicate Logic

Here are some logical symbols we will be using in this course, with their truth values (P and Q denote arbitrary statements):

- *Negation* ($\neg$, read "not"):

P	$\neg P$
T	F
F	T

- *Conjunction* ($\wedge$, read "and"):

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- *Disjunction* ($\vee$, read "or"):

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

- *Implication* ($\Rightarrow$, read "implies"):

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- *Equivalence* ($\Leftrightarrow$, read "if and only if"):

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Furthermore, there are two *quantifiers* that turn *predicates* into statements. Let $P(x)$ be a predicate involving an unknown variable x over a *universe of discourse* S (for example, $P(x)$: "$x > 0$", where the universe of discourse is the real numbers). The *universal quantifier*, denoted $\forall$ and read "for all", and the *existential quantifier*, denoted $\exists$ and read "there exists", may be used to turn this predicate into statements like $(\forall x \in S)[P(x)]$ or $(\exists x \in S)[P(x)]$.

Problem 1

For each of the following statements, translate it into logical symbols. Try to also figure out whether the statements are true or false.

1. For every real number, there exists a real number such that their product is 1.
2. For every natural number, there exists a natural number greater than it.
3. If the product of two integers is zero, one of them must be zero.
4. There is an integer that is its own square, and is not one or zero.
5. For any real numbers a, b, if $a - b$ is negative, then a must be greater than b.
6. An integer that is a perfect square and even must be divisible by 4.

To negate a statement, we may use the following:

(1) For any statement P, $\neg(\neg P)$ is equivalent to P.

(2) For any two statements P and Q, $\neg(P \vee Q)$ is equivalent to $\neg P \wedge \neg Q$, and $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$.[1] These are collectively referred to as *De Morgan's Laws*. Furthermore, $\neg(P \Rightarrow Q)$ is equivalent to $P \wedge \neg Q$.

(3) For any predicate $P(x)$ over a set A, $\neg(\forall x \in A)[P(x)]$ is equivalent to $(\exists x \in A)[\neg P(x)]$, and $\neg(\exists x \in A)[P(x)]$ is equivalent to $(\forall x \in A)[\neg P(x)]$.

Note that (3) may be used repeatedly for statements with multiple quantifiers:

$$\neg(\forall x \in A)(\exists y \in B)[P(x, y)] \Leftrightarrow (\exists x \in A)\neg(\exists y \in B)[P(x, y)] \Leftrightarrow (\exists x \in A)(\forall y \in B)[\neg P(x, y)].$$

For any statement P, either P is true, or $\neg P$ is true (and not both). We can show that an arbitrary statement P is false by showing that $\neg P$ is true. This is called a **proof by contradiction**.

Problem 2
For each of the following logic formulas, write down its negation in symbols. Determine whether it is true or its negation is true.

1. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})[m > n]$.
2. $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})[m > n]$.
3. $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})[m = 2n \Rightarrow (\forall \ell \in \mathbb{N})(m \neq 2\ell + 1)]$.
4. $(\forall q \in \mathbb{Q})(\exists m \in \mathbb{N})(\exists n \in \mathbb{N})[q = m/n]$.
5. $(\exists r \in \mathbb{Q})(\forall s \in \mathbb{Q})[s^2 \leq 2 \Rightarrow r \geq s]$.
6. $(\exists \epsilon > 0)(\forall \delta > 0)(\forall x \in \mathbb{R})[0 < |x - 2| < \epsilon \Rightarrow |3x - 6| < \delta]$.[a]
7. $(\exists \epsilon > 0)(\forall \delta > 0)(\forall x \in \mathbb{R})[0 < |x - 2| < \epsilon \Rightarrow 0 < 2]$.

[a] ϵ and δ are the Greek letters Epsilon and Delta respectively. These two letters will reappear throughout the course and haunt you in your dreams.

Problem 3
Show that the following are equivalent, no matter the truth values of P and Q, by enumerating through all four combinations of truth values of P and Q and computing the truth value on both sides of $\Leftrightarrow$:

1. $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.
2. $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$.

Problem 4
For each of the following, give a definition in symbols.

1. The integer n is divisible by the integer m.
2. The real number r is rational.
3. The integer d is the greatest common divisor of the integers m and n.
4. The set S has a maximum.

[1] $\neg$ takes precedence over $\wedge$ and $\vee$.

Sets and Cardinality

Given sets A, B (contained within some universe of discourse U), we say $A \subseteq B$ ("A is a subset of B") when every element of A is also an element of B. In other words,

$$(A \subseteq B) \Leftrightarrow (\forall x \in A)[x \in B].$$

We say that $A = B$ if both $A \subseteq B$ and $B \subseteq A$.

Also recall the set operations:

$$A \cup B = \{x \in U : x \in A \vee x \in B\}.$$

$$A \cap B = \{x \in U : x \in A \wedge x \in B\}.$$

$$A \setminus B = \{x \in U : x \in A \wedge x \notin B\}.$$

$$A^C = \{x \in U : x \notin A\}.$$

Problem 5
Let A, B, C be contained in some universe of discourse U. Prove the following statements:

1. If $A \subseteq B$ then $B^C \subseteq A^C$.
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
3. $(A^C)^C = A$.
4. $A \setminus B = A \cap B^C$.

Problem 6
Answer the following questions. Give a brief argument for your answer:

1. If A and B are finite sets, what is $|A \times B|$?
2. If A and B are countable sets, is $A \times B$ countable? If it's true in general, explain why. If it's not, come up with countable A and B so that $A \times B$ is not countable.
3. If A and B are finite sets, and $\mathcal{F}$ is the set of all functions $A \to B$, what is $|\mathcal{F}|$?
4. If $J_n = \{1, 2, 3, \ldots, n\}$ and $\mathcal{G}$ is the set of all *bijective* functions $J_n \to J_n$, what is $|\mathcal{G}|$?
5. Define $\mathcal{N} = \{S \subseteq \mathbb{N} : S \text{ is finite}\}$. In other words, $\mathcal{N}$ is the set of finite subsets of $\mathbb{N}$. Is $\mathcal{N}$ countable? Can you construct an injection from $\mathcal{N}$ into $\mathbb{N}$?